JOURNAL OF APPROXIMATION THEORY **67**, 235–238 (1991)

Note

A Gauss–Lucas Type Theorem on the Location of the Roots of a Polynomial

M. D. BUHMANN

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England, and Mathematical Sciences Department, IBM T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598, U.S.A.

AND

T. J. RIVLIN

Mathematical Sciences Department, IBM T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598, U.S.A.

Communicated by Allan Pinkus

Received November 19, 1990

In this note, we prove a geometrical relationship between the zeros of a polynomial p of order m, say, and the zeros of another polynomial which is derived from p by multiplying each of p's coefficients, call them $\{\alpha_k\}_{k=0}^m$, by a power of k or by $k^2+2k\lambda$ for $\lambda>0$.

The Gauss–Lucas Theorem (see, for instance, [1]) states that the zeros of the derivative of a polynomial have to lie in the convex hull of the zeros of the polynomial itself. In this note we establish a similar relationship between the zeros of a polynomial p of degree m which is expressed as a linear combination of certain basis polynomials φ_k, with φ_k of degree k, $k=0, 1, ..., m$, that span the space of all polynomials of degree m, and the zeros of a polynomial q which is obtained from p by multiplying the coefficient of φ_k by k^2 for all k, or by $k^2+2k\lambda$ for positive λ for all k.

When the basis functions are the Chebyshev polynomials, a result, which is also useful for relating the zeros of the second derivative of an even trigonometric polynomial to the zeros of the polynomial itself, is the following.

0021-9045/91 $3.00

THEOREM 1. *Let* $p=\sum_{k=0}^{m}\alpha_k T_k$ *be a polynomial, written in the Chebyshev basis, and let* $q=\sum_{k=0}^{m}\alpha_k k^2 T_k$. *Then, if* 0, 1, *or* -1, *is a zero of* q, *it has to lie in the convex hull* $\mathscr{P}'$ *of the zeros of* p', *and therefore in the convex hull* $\mathscr{P}$ *of the zeros of* p. *If* $x_0 \notin \{0, 1, -1\}$ *is a zero of* q, *it has to lie in the convex hull of* $\mathscr{P}' \cup \{x_0^{-1}\}$, *and therefore in the convex hull of* $\mathscr{P} \cup \{x_0^{-1}\}$.

Remark 1. In many cases, the requirement that x_0 be in the convex hull of $\mathscr{P}' \cup \{x_0^{-1}\}$ already means that x_0 has to be in $\mathscr{P}'$, e.g., if

(i) $x_0^{-1} \in \mathscr{P}'$, or

(ii) the line segment connecting x_0 and x_0^{-1} intersects $\mathscr{P}'$, or

(iii) the line through x_0 and x_0^{-1} does not intersect $\mathscr{P}'$, or

(iv) the ray from x_0 through x_0^{-1} intersects $\mathscr{P}'$.

Proof. We note first that by the Gauss–Lucas Theorem $\mathscr{P}' \subset \mathscr{P}$. (This fact has already been used twice in the statement of the theorem.) Now, by the differential equation

$$(1-x^2)\, T_k''(x) - xT_k'(x) + k^2 T_k(x) = 0$$

which is satisfied by the Chebyshev polynomials, it is true that

$$\sum_{k=0}^{m} \alpha_k k^2 T_k(x) = (x^2-1) \sum_{k=0}^{m} \alpha_k T_k''(x) + x \sum_{k=0}^{m} \alpha_k T_k'(x),$$

and we therefore have

$$q(x) = (x^2-1)\, p''(x) + xp'(x). \tag{1}$$

Suppose that $x_0 = 0$ is a zero of q. Then, by (1), $p''(x_0)=0$, whence 0 is in the convex hull of the zeros of p'', and therefore it is in $\mathscr{P}'$, by the Gauss–Lucas Theorem, as required. Suppose that $x_0 = 1$ or $x_0 = -1$ is a zero of q. Then, again by (1), $p'(x_0)=0$, which implies $x_0 \in \mathscr{P}'$. In all other cases, $q(x_0)=0$ implies

$$\frac{p''(x_0)}{p'(x_0)} = \frac{x_0}{1-x_0^2}, \tag{2}$$

where we assume that x_0 is not already a zero of p', because in that case the result follows immediately. We can rewrite (2) as

$$\sum_{j=1}^{m-1} \frac{1}{x_0 - x_j} = \frac{x_0}{1-x_0^2},$$

where $\{x_j \mid 1 \leqslant j \leqslant m-1\}$ are the zeros of p' (multiple zeros being counted multiply). Hence,

$$\sum_{j=1}^{m-1} \frac{\overline{x_0 - x_j}}{|x_0 - x_j|^2} = \frac{1}{x_0^{-1} - x_0} = \frac{\overline{x_0^{-1} - x_0}}{|x_0 - x_0^{-1}|^2},$$

and therefore

$$\sum_{j=1}^{m} \frac{x_0 - x_j}{|x_0 - x_j|^2} = 0, \tag{3}$$

where we let $x_m := x_0^{-1}$. Now let us define

$$\mu_j := \frac{|x_0 - x_j|^{-2}}{\sum_{l=1}^{m} |x_0 - x_l|^{-2}}$$

for all $1 \leqslant j \leqslant m$. Then (3) implies

$$x_0 = \sum_{j=1}^{m} \mu_j x_j, \tag{4}$$

where

$$\sum_{j=1}^{m} \mu_j = 1 \qquad \text{and} \qquad \mu_j > 0 \qquad \text{for all } j. \tag{5}$$

Expressions (4) and (5) imply the theorem. ∎

Remark 2. The assertion of the theorem remains true if we replace Chebyshev polynomials by any ultraspherical polynomials $P_k^{(\lambda)}$, where q now becomes $q = \sum_{k=0}^{m} \alpha_k k(k+2\lambda) P_k^{(\lambda)}$ and where λ is a positive constant.

COROLLARY. *Let p and q be as in the statement of the theorem or of Remark 2. Then the following statements are valid:*

(i) *If all the roots of p are real, so are the roots of q.*

(ii) *If all the roots of p are in the upper (lower) half-plane, then so are the roots of q.*

(iii) *If all the roots of p are inside a closed disk $\mathscr{D}$ about the origin of radius $r \geqslant 1$, so are the roots of q.*

Proof. We prove (i): If the roots of p are real, then $\mathscr{P}'$ is a subset of the real line. Suppose $q(x_0) = 0$. If x_0 is real, we are done. Otherwise x_0^{-1} lies in the other half-plane than x_0, i.e., the imaginary parts of x_0 and x_0^{-1} have opposite signs, thus contradicting the theorem. The second claim is established in a similar way as is the first one. We prove the last claim. Suppose x_0 is a root of q. If it is inside the closed disk $\mathscr{D}$, there is nothing to prove. Otherwise, x_0^{-1} will be inside $\mathscr{D}$, and so x_0 cannot be in the convex hull of $\mathscr{P}' \cup \{x_0^{-1}\}$, thus contradicting the assertion of our theorem. The corollary is proved. ▮

In case p is expressed as a linear combination of monomials, which can be considered as the limiting case of the one studied in Remark 2 for $\lambda \to \infty$, we have the following result.

THEOREM 2. *Let* $p(x) = \sum_{k=0}^{m} \alpha_k x^k$ *and* $q_n(x) = \sum_{k=0}^{m} \alpha_k k^n x^k$ *for a positive integer n. Then all zeros of* q_n *lie in the convex hull of* $\mathscr{P}' \cup \{0\}$.

Proof. We argue inductively, using the simple identity

$$q_n(x) = x q'_{n-1}(x) \tag{6}$$

which is true for positive n. For $n = 1$, the assertion of the theorem follows directly from (6) because $q_0 = p$ and therefore $q'_0 = p'$. If the assertion is true for q_{n-1}, then (6) and the Gauss–Lucas theorem imply that it also holds for q_n. The theorem is proved. ▮

REFERENCE

1. G. PÓLYA AND G. SZEGŐ, "Problems and Theorems in Analysis," Vol. 1, p. 108, Springer-Verlag, Berlin/Heidelberg/New York, 1972.

Printed in Belgium